

Hyperbolic Dehn surgery on geometrically infinite 3-manifolds

K. Bromberg*

July 24th, 2000

Abstract

In this paper we extend Thurston's hyperbolic Dehn surgery theorem to a class of geometrically infinite hyperbolic 3-manifolds. As an application we prove a modest density theorem for Kleinian groups. We also discuss hyperbolic Dehn surgery on geometrically finite hyperbolic cone-manifolds.

1 Introduction

An essential result in the theory of hyperbolic 3-manifolds is Thurston's Dehn surgery theorem. It states that all but a finite number of Dehn fillings on a hyperbolic knot complement have a hyperbolic structure. This theorem along with its extension to geometrically finite manifolds has strong geometric and topological consequences. In this paper we extend the Dehn surgery theorem to a large class of geometrically infinite hyperbolic manifolds. As an application we show that such geometrically infinite manifolds with rank one cusps can be approximated by manifolds without rank one cusps.

The Dehn filling theorem first appeared in Thurston's notes, [Th]. A number of the pieces of the argument have been published in a more rigorous form. We will indicate references as we outline the result. There is also an approach to the finite volume Dehn filling theorem via ideal triangulations. See [PP].

In the simplest case, M is a hyperbolic structure on the interior of a compact manifold with boundary consisting of a single torus. This hyperbolic structure defines a holonomy representation, ρ , of $\pi_1(M)$ in $PSL_2\mathbb{C}$.

*This work was partially supported by a grant from the Rackham School of Graduate Studies, University of Michigan and partially conducted for the Clay Mathematics Institute

The first step is to show that the space of all representations, $R(M)$, has complex dimension ≥ 1 at ρ (see [CS]).

The complex length of an element in $PSL_2\mathbb{C}$ measures the translation length and rotation of the isometry. For an element in $\pi_1(M)$ the complex length is a holomorphic function on $R(M)$. The complex length of a curve on the torus boundary is zero at any complete structure. Therefore by Mostow-Prasad rigidity this holomorphic function is not constant and by the dimension count for $R(M)$ must be locally onto.

To turn algebra into geometry, Thurston proves a result that originated in work of Weil, [We1]. Namely, if N is a compact hyperbolic manifold with holonomy ρ , and ρ' is a representation near ρ , then there is a nearby hyperbolic structure, N' , with holonomy ρ' . Furthermore, if N'' is another nearby structure with holonomy ρ' , then there is a slightly larger hyperbolic structure in which both N' and N'' isometrically embed. A more careful statement and proof can be found in [CEG].

For the Dehn filling theorem we let N be a compact core of our finite volume cusped manifold, M , and apply the above result. By the algebraic result we know that the representation deforms and this in turn gives a new hyperbolic structure, N' , on the compact core. The above deformation theorem does not apply to the non-compact cusp. Instead the geometry of the torus end must be constructed by hand.

The metric completion of the structures we build on the torus ends is described by a pair of real numbers, (a, b) , the *hyperbolic Dehn filling coefficient*. If a/b is rational then the metric completion will consist of adding a simple close curved. The topology of the completion will then be a solid torus. Let (p, q) be a relatively prime pair of integers such that $a/b = p/q$. Then the (p, q) -curve on the torus will be trivial in the solid torus. The metric around the added curve will usually be singular with cross section a hyperbolic cone. If $p = a$ and $q = b$ then the metric will be smooth.

Once the structures on the torus end have been constructed we need a gluing theorem to show that the hyperbolic structure, N' , on the compact core can be extended to one of these structures. This part of Thurston's argument has not been published. Although not difficult, the application of this result to geometrically infinite ends gave the the author enough pause that he felt it should be written down in detail.

If we assign the complete, cusped structure the Dehn filling coefficient ∞ , then the map to the space of Dehn filling coefficients, $\mathbb{R}^2 \cup \infty$, is continuous. The final step in the Dehn filling theorem is to show that if the complex length functions for curves on the torus boundary is locally onto, then the map to Dehn filling coefficients is locally onto. Since all but a finite number

of relatively prime pairs are in a neighborhood of ∞ this completes our outline.

We now assume that M is a complete hyperbolic structure on the interior of a compact manifold with boundary consisting of two components, a torus and a surface of higher genus, S . We also assume that this structure is geometrically finite without rank one cusps. In this case $R(M)$ will have complex dimension $\geq 1 + \dim T(S)$ where $T(S)$ is the Teichmüller space of S . The dimension of the space of complete structures on M is equal to the dimension of $T(S)$, so once again the map to Dehn filling coefficients is locally onto.

Every Dehn filling coefficient near ∞ is realized by a representation ρ' near ρ . Once again this gives a hyperbolic structure on a compact core N' . This structure extends at the torus end just as before. At the higher genus end we wish to extend N' to a geometrically finite structure. This is possible because geometrically finite ends have a strong stability property: For any small perturbation of the representation of a geometrically finite end, there is a nearby geometrically finite end with that representation. Then Dehn filling theorem for geometrically finite manifolds is proved in [Com]. Also see [BO].

In [HK], Hodgson and Kerckhoff extend the Dehn filling theorem to cone-manifolds. In particular, they prove that finite volume cone-manifolds are locally rigid when all cone angles are $\leq 2\pi$. Furthermore, they show that local rigidity implies that the map to Dehn filling coefficients is a local homeomorphism, not just locally onto. This is extended to geometrically finite cone manifolds in [Br]. This algebraic result allows us to prove the following strong Dehn filling theorem: Geometrically finite cone-manifolds are locally parameterized by Dehn filling coefficients and the conformal structure at infinity if all cone angles are $\leq 2\pi$. The complete case corresponds to cone angle 0.

The main focus of this paper is extending the Dehn filling theorem to geometrically infinite 3-manifolds. If M is geometrically infinite many parts of the argument remain the same. The dimension count for $R(M)$ will still hold and the map to Dehn filling coefficients will still be onto. However, the geometrically infinite higher genus end will not be stable. To get around this problem we restrict to a class of structures where the geometrically infinite end will be semi-stable. If none of the curves on the boundary torus of M are homotopic to the geometrically infinite end then there will be a half dimensional subspace of the representation variety for S , where the infinite end will be stable. What we will see is that for every Dehn filling coefficient near ∞ there will be a representation that restricted to the geometrically

infinite will be in the stable subspace.

The main application of the geometrically infinite Dehn filling theorem will be an approximation theorem for Kleinian groups. As an example let Γ be a Kleinian group such that $\mathbb{H}^3/\Gamma = M$ is homeomorphic to the interior of a manifold with two incompressible boundary components. We assume that one of the ends of M is geometrically infinite while the other is geometrically finite *with* rank one cusps. Our approximation theorem will show that we can resolve the rank one cusps. That is we can find Kleinian groups, Γ_i , and hyperbolic 3-manifolds, $M_i = \mathbb{H}^3/\Gamma_i$, such that $\Gamma_i \rightarrow \Gamma$ and M_i is homeomorphic to M with one geometrically infinite quasi-isometric to the infinite end of M and the other end geometrically finite *without* rank one cusps. This is a very special case of the density conjecture of Bers and Thurston.

In §2 we collect the background that we will need. In §3, we prove the algebraic piece of the geometrically infinite Dehn filling theorem. Then next two sections are expositions of material than can be found in Thurston's notes: In §4 we prove the gluing theorem and in §5 we construct the structures on the torus ends. For the Dehn filling theorems for infinite volume hyperbolic manifolds we need to construct structures on higher genus ends. This is done in §6. We put everything together to prove the Dehn filling theorems in §7. We prove the approximation theorem in §8.

Acknowledgments. The author would like to thank Dick Canary for many helpful discussions.

2 Preliminaries

We begin with a review of some basic facts about hyperbolic 3-manifolds.

2.1 Kleinian groups

Hyperbolic 3-space, \mathbb{H}^3 , is the unique simply connected 3-manifold with constant sectional curvature equal to -1 . The group of orientation preserving isometry groups of \mathbb{H}^3 is naturally isomorphic to $PSL_2\mathbb{C}$. Furthermore \mathbb{H}^3 can be compactified by the Riemann sphere, $\widehat{\mathbb{C}}$, and the isometric action of elements of $PSL_2\mathbb{C}$ on \mathbb{H}^3 extends continuously to projective transformations of $\widehat{\mathbb{C}}$.

In this paper, a *Kleinian group*, Γ , is a discrete, torsion free subgroup of $PSL_2\mathbb{C}$. Γ will act properly discontinuously on \mathbb{H}^3 so the quotient, $M = \mathbb{H}^3/\Gamma$, will be a hyperbolic 3-manifold.

2.2 The domain of discontinuity and limit set

The action of Γ on $\widehat{\mathbb{C}}$ naturally decomposes $\widehat{\mathbb{C}}$ into two sets. The *domain of discontinuity*, $\Omega(\Gamma)$, is the largest subset of $\widehat{\mathbb{C}}$ where Γ acts properly discontinuously. The *limit set*, $\Lambda(\Gamma)$, is the complement of $\Omega(\Gamma)$. By Ahlfors finiteness theorem, if Γ is finitely generated, $\Omega(\Gamma)/\Gamma$ is a finite collection of Riemann surfaces of finite type. As we shall see $\Omega(\Gamma)/\Gamma$ can be naturally identified with the quotient hyperbolic 3-manifold to form a (not necessarily compact) 3-manifold with boundary.

2.3 Laminations

There are a number of ways to define a lamination on a surface S . The easiest is to endow S with a hyperbolic metric. Then a *lamination* on S is a collection of disjoint, simple geodesics whose union is a closed subset of S . However, the use of the hyperbolic metric is unnecessary and with more work we could define this concept pure topologically.

A lamination, λ , is finite if it consists entirely of simple closed curves on S . In general, λ , will be a limit of finite laminations in the Hausdorff topology on closed subsets of S .

2.4 Ends

Throughout this paper we will assume that all 3-manifolds are tame. That is, M will be the interior of a compact 3-manifold with boundary, N . If S is a connected component of ∂N then a neighborhood E of S is an *end* of N . In abuse of notation E will also refer to the restriction of this neighborhood to M . We can assume that E is an I -bundle over S . In particular we identify E with $S \times [0, 1]$ with $S \times 1$ lying in ∂N .

A sequence of simple closed curves, c_n , *exits* E if for every $t \in [0, 1)$, c_n is contained in $S \times [t, 1)$ for all large n .

2.5 Ending laminations

A complete hyperbolic structure on M determines a lamination, λ , on each component S of ∂N . Loosely speaking λ will be the collection of geodesics which exit the end E . We make a precise definition of λ by examining its connected components.

If a connected component, c , of λ is a simple closed curve then there will be simple closed curves c_i exiting E each isotopic to c in N such that the length of the c_i in M goes to zero.

For a connected component, λ_0 , of λ that is not a simple closed curve there will be a sequence of simple closed curves c_i on S converging to λ_0 such that the geodesic representatives of the c_i exit E . We say that c is a *rank-one cusp* of E .

The union of all such laminations satisfying these two conditions is the *ending lamination*, λ , for E . For a tame end the work of Bonahon and Canary shows that λ is well defined and unique. The ending lamination for M is the union of the ending laminations for each end.

2.6 The conformal structure at infinity

The Kleinian manifold \hat{M} is the quotient of $\mathbb{H}^3 \cup \Omega(\Gamma)$ under the action of Γ . The inclusion of M into N will extend to an inclusion of \hat{M} into N . The image of \hat{M} in ∂N will be a collection of essential subsurfaces bounded by simple closed curves in the ending lamination. Since $\Omega(\Gamma)/\Gamma$ is a Riemann surface we have defined a *conformal structure at infinity* on a subsurface of ∂N .

2.7 End-invariants

We have now defined two objects on ∂N : the ending lamination, λ , and the conformal structure at infinity, μ . The key fact is that these two objects will have disjoint support. Furthermore λ will be a maximal in the sense that any simple closed curve disjoint from λ and the support of μ will be homotopically trivial. The *end-invariant* of a complete hyperbolic structure on M is the pair (λ, μ) . Thurston's ending lamination conjecture states that this pair uniquely determines the hyperbolic structure on M .

2.8 Accidental parabolics

If E is the end associated to the component S of ∂N then c is an *accidental parabolic* if c has parabolic holonomy yet is not in the ending lamination of E . As a consequence of negative curvature, two simple closed curves that are not homotopic on ∂N but are homotopic in N cannot both be in the ending lamination. Therefore if one of the curves is in the ending lamination the other will be an accidental parabolic.

2.9 Geometrically finite and infinite hyperbolic structures

If an end has empty ending lamination then it is *geometrically finite without rank-one cusps*. If the ending lamination is a finite lamination then the end is

simply *geometrically finite*. Otherwise the end is *geometrically infinite*. We make similar definitions for hyperbolic structures on the entire manifold, M .

2.10 Quasiconformal deformation spaces

A *quasiconformal homeomorphism* between two Riemann surfaces is a homeomorphism, f , with distributional derivatives locally in L^2 and $\|f_{\bar{z}}/f_z\|_\infty < 1$. The quantity, $\mu = f_{\bar{z}}/f_z$ is the *beltrami differential* for f and is a differential of type $(-1, 1)$. The map f is k -quasiconformal if $k = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}$.

Let Γ and Γ' be two Kleinian groups with a group isomorphism taking $\gamma \in \Gamma$ to $\gamma' \in \Gamma'$. Γ' is a *quasiconformal deformation* of Γ if there is a quasiconformal homeomorphism, $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, such that $f \circ \gamma = \gamma' \circ f$ for all $\gamma \in \Gamma$. Let $\mathcal{QD}(\Gamma)$ be the space of Kleinian groups which are quasiconformal deformations of Γ and $QD(\Gamma)$ be the quotient of $\mathcal{QD}(\Gamma)$ under the action of $PSL_2\mathbb{C}$ by conjugation.

Assume $M = \mathbb{H}^3/\Gamma$ is tame and let S be a subsurface of ∂M bounded by simple closed curves in the ending lamination. Then a component of $\Omega(\Gamma)/\Gamma$ lie in S in we let $\Omega_S \subseteq \Omega(\Gamma)$ be the pre-image of this component. Then $QD(\Gamma; S) \subseteq QD(\Gamma)$ is the space of quasiconformal deformations such that there exist a conjugation quasiconformal homeomorphism whose beltrami differential's support is disjoint from Ω_S .

It will often be convenient to refer to the elements of $QD(\Gamma)$ which are equivalence classes of groups by a single group in the class. We define $\Gamma' \in QD(\Gamma)$ to be the unique group in it's equivalence class that is the conjugation of Γ by a quasiconformal homeomorphism fixing 0, 1 and ∞ . This also gives a canonical identification of $\mathcal{QD}(\Gamma)$ with $QD(\Gamma) \times PSL_2\mathbb{C}$.

2.11 Teichmüller spaces

Let S be a closed surface with a finite number of punctures. The *Teichmüller space* of S , $T(S)$, is the space marked, complex structures on S or equivalently the space of marked, complete, finite area hyperbolic structures on S . $T(S)$ has a canonical complex structure and is a cell of complex dimension $3g - 3 + p$ where g is the genus of S and p the number of punctures.

The following theorem is due to Ahlfors, Bers, Kra, Maskit and Sullivan.

Theorem 2.1 *$QD(\Gamma)$ is locally parameterized by $T(\Omega(\Gamma)/\Gamma)$. $QD(\Gamma; S)$ is locally parameterized by $T(\Omega_C/\Gamma)$ where Ω_C is the complement of Ω_S in $\Omega(\Gamma)$.*

This parameterization is global if \mathbb{H}^3/Γ has incompressible boundary. There is also a global parameterization for the general case although it is more difficult to state and we will not need it.

Note that the complex structure on Teichmüller space defines a complex structure on $QD(\Gamma)$.

2.12 Representation varieties

We let $\mathcal{R}(\Gamma)$ be the space all representations of the abstract group Γ in $PSL_2\mathbb{C}$. Since $PSL_2\mathbb{C}$ is a complex, algebraic group, $\mathcal{R}(\Gamma)$ has a naturally structure as a complex variety. We let $R(\Gamma)$ be the quotient of $\mathcal{R}(\Gamma)$ under the action of $PSL_2\mathbb{C}$. If Γ is the fundamental group of a manifold M , $\mathcal{R}(M)$ and $R(M)$ will be representations of $\pi_1(M)$ and the quotient space, respectively.

There is a natural inclusions of $QD(\Gamma)$ in $R(\Gamma)$. The Ahlfors-Bers version of the measurable Reimann mapping theorem implies that this inclusion is holomorphic. Kapovich has shown that $R(\Gamma)$ is a smooth variety at all discrete faithful representations. In particular, $R(\Gamma)$ is a smooth complex manifold in a neighborhood of the image of $QD(\Gamma)$. Therefore we can view the inclusion map as a holomorphic embedding of one complex manifold in another.

2.13 Group cohomology

A *1-cocycle* is a map, $z : \Gamma \rightarrow sl_2\mathbb{C}$ with $z(\gamma_1\gamma_2) = z(\gamma_1) + \text{Ad } \rho(\gamma_1)z(\gamma_2)$. A 1-cocycle is a *1-coboundary* if there exists a $v \in sl_2\mathbb{C}$ such that $z(\gamma) = v - \text{Ad } \rho(\gamma)v$ for all $\gamma \in \Gamma$. We let $Z^1(\Gamma; \text{Ad } \rho)$ be the space of 1-cocycles, $B^1(\Gamma; \text{Ad } \rho)$ the space of 1-coboundaries, and

$$H^1(\Gamma; \text{Ad } \rho) = Z^1(\Gamma; \text{Ad } \rho)/B^1(\Gamma; \text{Ad } \rho).$$

We then have:

Theorem 2.2 ([We2]) *$H^1(\Gamma; \text{Ad } \rho)$ is canonically isomorphic to the Zariski tangent space, $TR(\Gamma)$, of $R(\Gamma)$ at Γ .*

Remark. Strictly speaking, for this theorem to be true, $H^1(\Gamma; \text{Ad } \rho)$ should be viewed as the Zariski tangent space of the scheme, $R(\Gamma)$. However, at the discrete faithful representations, Kapovich's theorem, mentioned above, implies that $R(\Gamma)$ is a smooth complex manifold and the Zariski tangent space can be canonically identified with the differentiable tangent space.

2.14 Strain fields

For a vector field on $\widehat{\mathbb{C}}$ the *strain* is the infinitesimal version of the beltrami differential of a homeomorphism. As we will also need the notion of the strain of a vector field on \mathbb{H}^3 , we define the strain, Sv , of a general vector field, v , on a conformally flat manifold as a tensor of type (1,1) with

$$S_{ij}v = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{\delta_j^i}{n} \sum_{i=1}^n \frac{\partial v_i}{\partial x_i}$$

where $v = (v_1, \dots, v_n)$ on a local, conformal chart. If $v = f \frac{\partial}{\partial z}$ is a vector field on $\widehat{\mathbb{C}}$ an easy calculation shows that $Sv = f \frac{d\bar{z}}{dz}$. We also let $|Sv|$ be the operator norm of Sv . Then v is a *quasiconformal vector field* if Sv is measurable and $|Sv|$ is essentially bounded.

2.15 Infinitesimal deformations

An *infinitesimal quasiconformal deformation* of Γ is a quasiconformal vector field, v , on $\widehat{\mathbb{C}}$ such that there exists a map, $z : \Gamma \rightarrow sl_2\mathbb{C}$ with $v - \gamma_*v = z(\gamma)$. By the chain rule z is cocycle. An *infinitesimal hyperbolic deformation* of Γ is a smooth vector field, v , on \mathbb{H}^3 such that $v - \gamma_*v = z(\gamma)$.

Note that both types of infinitesimal deformations form a vector space and there is a homomorphism from each of these vector spaces to $H^1(\Gamma; \text{Ad } \rho)$. For infinitesimal hyperbolic deformations this map will be onto. In particular, we can define v to be zero on a fundamental domain for Γ in \mathbb{H}^3 and use the cocycle condition to define v on all of \mathbb{H}^3 . The vector field, v , can be made smooth via a partition of unity argument.

On the other hand the map from infinitesimal quasiconformal deformations to $H^1(\Gamma; \text{Ad } \rho)$ will not always be onto. Let $TQD(\Gamma)$ be the subspace of $TR(\Gamma)$ which is the tangent space of the embedding of $QD(\Gamma)$ in $R(\Gamma)$. We then have the following theorem which can be found in [Ah].

Theorem 2.3 *Let $z \in TQD(\Gamma)$ be a cocycle. Then there exists an infinitesimal quasiconformal deformation, v , such that $v - \gamma_*v = z(\gamma)$.*

2.16 Continuous extension of vector fields on \mathbb{H}^3 to $\widehat{\mathbb{C}}$

If we view \mathbb{H}^3 as the open unit ball in \mathbb{R}^3 and $\widehat{\mathbb{C}}$ as the unit sphere then we can discuss the continuous extension of vector fields on \mathbb{H}^3 to $\widehat{\mathbb{C}}$. A general vector field on \mathbb{H}^3 will not continuously extend to $\widehat{\mathbb{C}}$, and even if the vector field in the unit ball does extend to the unit sphere, it may not be tangent

to the unit sphere and therefore will not be a continuous vector field on $\widehat{\mathbb{C}}$. This leads us to consider the weaker notion of a continuous *tangential* extension. Given a vector field v on \mathbb{H}^3 and a vector field V on $\widehat{\mathbb{C}}$ we say that V is a continuous tangential extension of v if there exists a point $p \in \mathbb{H}^3$ such that the projection of v to the family of hyperbolic spheres centered at p extends continuously to $\widehat{\mathbb{C}}$. Note that the choice of p is arbitrary and if such a continuous extension exists for a single point, $p \in \mathbb{H}^3$, then it will exist for every point in \mathbb{H}^3 and the extension to $\widehat{\mathbb{C}}$ will always be the same vector field.

The following lemma is an easy exercise:

Lemma 2.4 *If v is an infinitesimal hyperbolic deformation and V is a continuous tangential extension of v that is a quasiconformal vector field then V will be an infinitesimal quasiconformal deformation that determines the same cocycle as v .*

2.17 Averaging vector fields on $\widehat{\mathbb{C}}$

Let $V(\widehat{\mathbb{C}})$ and $V(\mathbb{H}^3)$ be the space of (distributional) vector fields on $\widehat{\mathbb{C}}$ and \mathbb{H}^3 respectively. The following theorem is due to Ahlfors, Reimann and Thurston. A self-contained exposition can be found in [Mc].

Theorem 2.5 *There exists a unique operator, $\text{ex} : V(\widehat{\mathbb{C}}) \longrightarrow V(\mathbb{H}^3)$, such that for every $v \in V(\widehat{\mathbb{C}})$ the following properties hold:*

1. $\text{ex}(v)$ is smooth and extends continuously to v if v is continuous.
2. If v is quasiconformal, $\text{ex}(v)$ is quasiconformal.
3. For every $\gamma \in PSL_2\mathbb{C}$, $\gamma_* \text{ex}(v) = \text{ex}(\gamma_* v)$.
4. If v is a projective vector field, $\text{ex}(v)$ is an infinitesimal isometry.

The vector field $\text{ex}(v)$ evaluated at x is essentially the the *visual average* of v when viewed from x in hyperbolic space.

3 Geometrically infinite manifolds

Let N be a compact, hyperbolizable 3-manifold with interior M . Assume N is bounded by k tori and $n - k$ surfaces of genus > 1 with $n > k > 1$. Label the tori T_1, \dots, T_k and the surfaces of higher genus S_{k+1}, \dots, S_n and let E_i be the associated ends. Let Γ be a Kleinian group such that $M = \mathbb{H}^3/\Gamma$.

Let \tilde{E}_i be a lift of E_i to the universal cover, \tilde{M} , and $\Gamma_i \subset \Gamma$ the subgroup fixing \tilde{E}_i . Choose an integer m , such that $k < m \leq n$. Throughout this section we make the following assumptions:

1. The surfaces S_i are incompressible for $k < i \leq m$.
2. E_i does not have accidental parabolics for $k < i \leq m$.
3. The hyperbolic structures on E_i are geometrically finite without rank one cusps for $i > m$.

Conditions (1) and (2) restrict the topological type of the manifold M along with the geometry. Although the class of manifolds that satisfy these two conditions is quite large, some of the simplest hyperbolic manifolds are not in it, such as I -bundles over a surface with a curve removed. The importance of these conditions becomes apparent in the following proposition.

Proposition 3.1 *The dimension of $QD(\Gamma_i; S_i)$ is equal to the dimension of the Teichmüller space of S_i for $i \leq m$.*

Proof. Since S_i is incompressible, Bonahon's theorem [Bon] implies that \mathbb{H}^3/Γ_i is homeomorphic to $S_i \times (0, 1)$. On the other hand, M is not an I -bundle because it has two non-homeomorphic boundary components. Therefore Γ_i has infinite index in Γ . We can assume E_i has been chosen such that the covering map is injective onto E_i . Let E_i^* be the other end of \mathbb{H}^3/Γ_i . The covering map from E_i^* to N must be infinite-to-one and therefore by the covering theorem [Can], E_i^* is geometrically finite.

Since E_i^* is geometrically finite its ending lamination consists entirely of simple closed curves. However, if c is a simple closed curve in the ending lamination for E_i^* then c is parabolic. However by assumption (3), c is then in the ending lamination for E_i . This is a contradiction so E_i^* has an empty ending lamination, and by Theorem 2.1, $QD(\Gamma_i; S_i)$ is locally parameterized by the Teichmüller space of S_i . 3.1

To simplify notation we let $\partial_1 M = S_{k+1} \cup \dots \cup S_m$ and $\partial_2 M = S_{m+1} \cup \dots \cup S_n$. $R(\partial_1 M)$, $R(\partial_2 M)$ and $T(\partial_2 M)$ are then the products of the respective representation varieties and Teichmüller spaces. $QD(\partial_1 M)$ is the product of relative quasiconformal deformation spaces $QD(\Gamma_i; S_i)$ with $i = k+1, \dots, m$. Let V be a small neighborhood ρ in $R(\Gamma)$. We then have restriction maps $\partial_i : R(\Gamma) \rightarrow R(\partial_i M)$ and for the geometrically finite ends a map to Teichmüller space, $b'_2 : V \rightarrow T(\partial_2 M)$ (see §6). For each torus boundary component a

choice of a simple closed curve determines a length function, $\mathcal{L}'_i : V \longrightarrow \mathbb{C}^k$ (see §5). Putting these maps together we have maps, $\partial : R(\Gamma) \longrightarrow \mathbb{C}^k \times R(\partial_1 M) \times R(\partial_2 M)$ defined by $\partial = (\mathcal{L}_1, \dots, \mathcal{L}_k, \partial_1, \partial_2)$ and $\Phi' : V \longrightarrow \mathbb{C}^k \times T(\partial_2 M)$ defined by $\Phi' = (\mathcal{L}'_1, \dots, \mathcal{L}'_k, b'_2)$.

Finally we define $CD(\partial_2 M) = (b'_2)^{-1}(b'_2(\partial_2(\Gamma)))$, the space of conformal deformations of the geometrically finite ends and $R(\Gamma; \partial_1 M) = \partial^{-1}(\mathbb{C}^k \times QD(\partial_1 M) \times R(\partial_2 M))$.

Theorem 3.2 Φ' restricted to $R(M; \partial_1 M)$ is a local homeomorphism at Γ .

Proof. We begin by noting the smoothness and dimension of the various spaces. All dimensions are complex. $R(\Gamma)$ is smooth at Γ and has dimension $-\frac{3}{2}\chi(\partial M) + k$ by Theorem 9.8.1 in [Kap2]. $R(\partial_1 M)$ and $R(\partial_2 M)$ are smooth at $\partial_1(\Gamma)$ and $\partial_2(\Gamma)$, respectively, with dimension $-3\chi(\partial_1 M)$ and $-3\chi(\partial_2 M)$, respectively [Gun]. $QD(\partial_1 M)$ is a complex submanifold of $R(\partial_1 M)$ and has dimension $-\frac{3}{2}\chi(\partial_1 M)$ by Lemma 3.1. $CD(\partial_2 M)$ is isomorphic to the space of holomorphic quadratic differentials on $\partial_2 M$ and therefore is smooth of dimension $-\frac{3}{2}\chi(\partial_2 M)$. Let $TR(\Gamma)$, $TR(\partial_1 M)$, $TR(\partial_2 M)$, TQD , and $TC D$ be the respective tangent spaces at Γ or its image under the appropriate map.

To prove the theorem we need to show that

$$\dim R(M; \partial_1 M) = \dim T(\partial_2 M) + k = -\frac{3}{2}\chi(\partial_2 M) + k$$

and that $(\Phi')_*$ is injective.

Assume that $v \in TR(\Gamma)$ is an infinitesimal hyperbolic deformation. We make the following claim: If $\partial_* v \in (0 \times TQD \times TC D)$ then v is a trivial deformation. We first will assume this claim and complete the proof of the theorem, saving the proof of the claim for below.

We first show that $R(M; \partial_1 M)$ is smooth and has the correct dimension by showing that $\partial_* TR(M)$ is transverse to $\mathbb{C}^k \times TQD \times TR(\partial_2 M)$ and that the dimension, D , of

$$\partial_* TR(\Gamma) \cap (\mathbb{C}^k \times TQD \times TR(\partial_2 M))$$

is $-\frac{3}{2}\chi(\partial M_2) + k$. The claim implies that ∂_* is injective and therefore

$$\dim \partial_* TR(\Gamma) = -\frac{3}{2}\chi(\partial M) + k$$

while

$$\begin{aligned} \dim(\mathbb{C}^k \times TQD \times TR(\partial_2 M)) &= -\frac{3}{2}\chi(\partial_1 M) - 3\chi(\partial_2 M) + k \\ &= -\frac{3}{2}\chi(\partial M) - \frac{3}{2}\chi(\partial_2 M) + k \end{aligned}$$

and

$$\begin{aligned}\dim(\mathbb{C}^k \times TR(\partial_1 M) \times TR(\partial_2 M)) &= -3\chi(\partial_1 M) - 3\chi(\partial_2 M) + k \\ &= -3\chi(\partial M) + k\end{aligned}$$

so

$$D \geq -\frac{3}{2}\chi(\partial_2 M) + k.$$

On the other hand, the claim also tells us that

$$\partial_* TR(M) \cap (0 \times TQD \times TCD) = 0$$

and since

$$\dim(\mathbb{C}^k \times TQD \times TCD) = -\frac{3}{2}\chi(\partial_1 M) - \frac{3}{2}\chi(\partial_2 M) + k,$$

we have

$$D \leq -\frac{3}{2}\chi(\partial_2 M) + k.$$

Therefore

$$D = -\frac{3}{2}\chi(\partial_2 M) + k.$$

Since the dimension of intersection is minimal, the intersection is transverse and therefore $R(\Gamma; \partial_1 M)$ is smooth at Γ and has dimension $D = \dim T(\partial_2 M) + k$.

We now show that $(\Phi')_*$ restricted to $TR(\Gamma; \partial_1 M)$ is injective. Since $R(\Gamma; \partial_1 M)$ is smooth we can discuss its tangent space, $TR(\Gamma; \partial_1 M)$, at Γ . If $v \in TR(\Gamma; \partial_1 M)$ then $\partial_* v \in (\mathbb{C}^k \times TQD \times TR(\partial_2 M))$ while if $(\Phi')_* v = 0$ then $\partial_* v \in (\mathbb{C}^k \times TQD \times TCD)$. By the claim, v is trivial, so the restriction of $(\Phi')_*$ to $TR(\Gamma; \partial_1 M)$ is injective and since $TR(\Gamma; \partial_1 M)$ and $\mathbb{C}^k \times T(\partial_2 M)$ have the same dimension, $(\Phi')_*$ is an isomorphism and therefore Φ' restricted to $R(\Gamma; \partial_1 M)$ is a local homeomorphism at Γ .

We now prove the claim in a sequence of two lemmas.

Lemma 3.3 *If $\partial_* v \in (0 \times TQD \times TCD)$ then v can be represented by a quasiconformal vector field that extends continuously to $\Omega(\Gamma)$.*

Proof. We first construct the vector field on the lift of the geometrically infinite ends. By assumption, $(\partial_1)_* v \in TQD$ so Theorem 2.3 implies that there exists a vector field V_i on $\widehat{\mathbb{C}}$ which is a infinitesimal quasiconformal deformation equivalent to v . We then let $v' = \text{ex}(V_i)$ on \tilde{E}_i . Theorem 2.5 implies that v' is quasiconformal on \tilde{E}_i . For $\gamma \in \Gamma$, define $v' = \gamma_* v' + v -$

γ_*v . It is easy to check that this is well defined using the fact that v is an infinitesimal deformation of Γ and v' is an infinitesimal deformation of Γ' . Furthermore, where it is defined, v' will be an infinitesimal hyperbolic deformation defining the same cocycle as v .

If $i > m$ we construct v' on \tilde{E}_i using Theorem 3.9 of [Br]. For the ends associated to the torus boundary components we use Proposition 3.10 of [Br].

The vector field, v' , is now defined outside of the universal cover of a compact subset of M , so as long as the extension of v' to the rest of \mathbb{H}^3 is smooth, v' will be quasiconformal. This extension can be done by combining v with v' using a cutoff function. We again refer to [Br] for details. 3.3

In the next lemma we show that this vector field extends to all of $\hat{\mathbb{C}}$.

Lemma 3.4 *Every quasiconformal vector field, $v \in TR(M)$, which extends continuously to $\Omega(\Gamma)$ has a continuous tangential extension to a quasiconformal vector field on all of $\hat{\mathbb{C}}$.*

Proof. If $\Lambda(\Gamma) = \hat{\mathbb{C}}$ then v has a continuous tangential extension on $\hat{\mathbb{C}}$ by Theorem 4.8 in [Kap1]. If not we must modify Kapovich's result only slightly.

Kapovich's proof has two steps:

1. The infinitesimal deformation, v , extends continuously to the loxodromic fixed points of Γ .
2. If a quasiconformal vector extends continuously to a dense subset of $\hat{\mathbb{C}}$ then it has a continuous tangential extension to a quasiconformal vector field on all of $\hat{\mathbb{C}}$.

If $\Lambda(\Gamma) = \hat{\mathbb{C}}$ then we follow Kapovich and apply (1). Since the loxodromic fixed points are dense in $\Lambda(\Gamma)$ this gives a continuous extension to a dense subset of $\hat{\mathbb{C}}$. Otherwise $\Omega(\Gamma)$ is dense in $\hat{\mathbb{C}}$ and we have assumed that v extends continuously to $\Omega(\Gamma)$. In both cases we then apply (2). 3.4

It is now an easy matter to complete the proof of the claim. By Lemmas 3.3 and 3.4 there exists a deformation, v' , equivalent to v that has a continuous tangential extension to a quasiconformal vector field, V , on $\hat{\mathbb{C}}$ that is conformal on $\Omega(\Gamma)$. Furthermore, by Lemma 2.4, V is an infinitesimal quasiconformal deformation that defines the same cocycle as v . Therefore SV will be an equivariant strain field which is zero on $\Omega(\Gamma)$ and by Sullivan's rigidity theorem, [Sul], SV is zero on all $\hat{\mathbb{C}}$. This implies that V is a conformal vector field and that the associated cocycle is a coboundary.

Hence, v is a trivial deformation proving the claim and Theorem 3.2. 3.2

4 Developing maps

A *geometry* is a pair made up of a simply connected manifold, X , with a real analytic, transitive group action G . The “geometry” of X may be a Riemannian metric with G the group of isometries. However, this is not the only possibility. For example G may be the group of projective or affine transformations of X .

For a manifold, M , a (G, X) -*developing map*, D , is a local homeomorphism from \tilde{M} to X such that there exists a *holonomy representation*, $\rho : \pi_1(M) \longrightarrow G$ with

$$D(\gamma(x)) = \rho(\gamma)(D(x))$$

for all $\gamma \in \pi_1(M)$ and $x \in \tilde{M}$. This defines a (G, X) -*structure* on M . We will denote the space of developing maps for M with the C^∞ topology by $\mathcal{D}(M)$.

M is a *thickening* of a compact manifold N if $M - N$ is homeomorphic to $\partial N \times (0, 1]$. The following is Theorem 1.7.1 in [CEG].

Theorem 4.1 *Let M be a thickening of N and D_0 a developing map for M with holonomy representation ρ_0 .*

1. *There exists a continuous map, D , from a neighborhood, V , of ρ_0 in $\mathcal{R}(M)$ to $\mathcal{D}(M)$ such that the holonomy of $D(\rho)$ is ρ .*
2. *For any V as in 1, there is a neighborhood of D_0 in $\mathcal{D}(N)$ that is homeomorphic to a product $\mathcal{I} \times V$, where \mathcal{I} is a neighborhood of the inclusion $N \hookrightarrow M$ in the space of locally flat embeddings. In particular, if $(\iota, \rho) \in \mathcal{I} \times V$ and $\tilde{\iota}$ is the lift of ι to the universal cover, then $(\iota, \rho) \mapsto D(\rho) \circ \tilde{\iota}$ under this homeomorphism.*

Let M be an open manifold and N_0, N_1, N_2 and N_3 submanifolds of M such that N_{i+1} is a thickening of N_i for $i = 0, 1, 2$ and M is the interior of a thickening of N_3 . Then *ends* of M will be the finite number of components of $M \setminus N_1$. Let E be an end of M . Finally, we define two more submanifolds of M ; C_1 is the closure of $(N_2 \setminus N_1) \cap E$ and C_2 is the closure of $(N_3 \setminus N_0) \cap E$.

Theorem 4.2 *Let D be a developing map for M and D_3 and D_E the restriction of D to N_3 and E , respectively. Assume that D'_3 and D'_E are small*

deformations of D_3 and D_E , respectively, such their restrictions to C_1 have the same holonomy. Then there exists a developing map, D' , for M such that $D'|_{N_0} = D'_3|_{N_0}$ and $D'|_E = D'_E$.

Proof. We apply Theorem 4.1 to C_1 and C_2 . Let ρ be the holonomy of

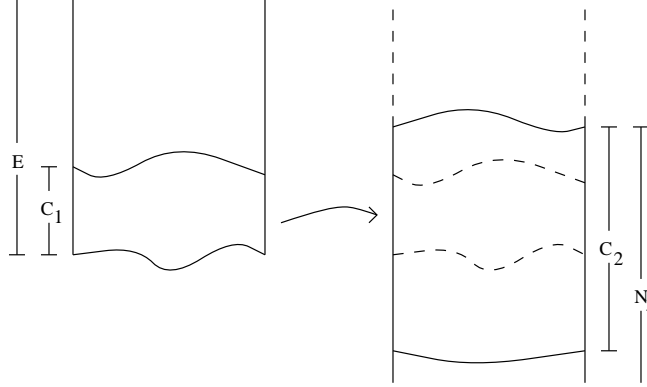


Figure 1: After we deform E and N_3 , the new structure on C_1 coming from E will embed in the new structure on C_2 coming from N_3 .

D restricted to C_2 . Then, as in (1), we have a map $D_C : V \rightarrow \mathcal{D}(C_2)$ where V is a neighborhood of ρ in $\mathcal{R}(C_2)$ and as in (2), we have a neighborhood of D restricted to C_1 parameterized by $V \times \mathcal{I}$ where \mathcal{I} is a neighborhood of the inclusion map in the space of locally flat embeddings.

Let ρ' be the holonomy of D_3 and D_E restricted to C_1 . By (2) there exist embeddings, $\iota_3, \iota_E : C_1 \rightarrow C_2$, such that $D'_3|_{C_1} = D_C(\rho') \circ \tilde{\iota}_3$ and $D'_E|_{C_1} = D_C(\rho') \circ \tilde{\iota}_E$. Extend ι_3 to a diffeomorphism, $\psi : N_3 \rightarrow N_3$ such that $\psi|_{N_0}$ is the identity. We can then find another diffeomorphism, $\phi : N_3 \rightarrow N_3$, such that $(\psi \circ \phi)|_{C_1} = \iota_E$. Define the developing map, D' , such that D' restricted to N_2 is $D'_3 \circ \tilde{\psi} \circ \tilde{\phi}$ and D' restricted to E is D'_E . Since $D'_E = D'_3 \circ \tilde{\psi} \circ \tilde{\phi}$ on C_1 , D' is well defined and is the desired developing map. 4.2

Using the developing map, we can pull back the geometric structure on X to a geometric structure on \tilde{M} . Since D is equivariant this geometric structure will descend to a geometric structure on M . We will refer to this structure as M_D to distinguish it from the differentiable manifold, M . We also define the diffeomorphism, $f_D : M \rightarrow M_D$. The pair, (M_D, f_D) , is *marked* geometric structure on M . (Often the developing map, D , will be

indexed by some i . In this case the associated structure is M_i and the diffeomorphism is f_i .) With this definition one defines marked geometric structures where, $(M_1, f_1) \sim (M_2, f_2)$ if there exists a geometry preserving homeomorphism, $f : M_1 \rightarrow M_2$, such that f is isotopic to $f_2 \circ f_1^{-1}$. The space of equivalence classes is the *Teichmüller space* of (G, X) -structures on M . This notion of Teichmüller space is a generalization of the Teichmüller space of complex structures discussed in §2.11.

Given a diffeomorphism, f , between two Riemannian manifolds, M_0 and M_1 we let $K(f)$ be the minimal biLipshitz constant of f . In general $K(f)$ may be infinite, although for compact manifolds it will always be finite. If there exists an f such that $K(f)$ is finite we say that M_0 and M_1 are *quasi-isometric*.

We next define geometric convergence. This concept is important because there are sequences of manifolds that are not quasi-isometric, yet are geometrically very close. Let (M_n, ω_n) be a sequence of Riemannian manifolds with basepoints. The sequence, (M_n, ω_n) , converges to $(M_\infty, \omega_\infty)$ *geometrically* if there exists an exhaustion of M_∞ by compact submanifolds, M_∞^n , and K_n -biLipshitz maps, $f^n : M_\infty^n \rightarrow M_n$, with $f^n(\omega) = \omega_n$ and $K_n \rightarrow 1$.

We then have the following theorem:

Proposition 4.3 *Let D_n be a sequence of developing maps such that $D_n \rightarrow D_\infty$ in $\mathcal{D}(M)$. Let $\omega_n = f_n(\omega)$ where $\omega \in M$. Then $(M_n, \omega_n) \rightarrow (M_\infty, \omega_\infty)$ geometrically.*

Proof. First we assume that M is compact. Let V be a neighborhood of $x \in \tilde{M}$ such that D_∞ restricted to V is a homeomorphism onto its image. Then for large n , D_n restricted to V will be a homeomorphism. Since D_n converges to D_∞ in the C^∞ -topology, $D_n \circ D_\infty^{-1}$ restricted to $D_\infty(V)$ converges to the identity in the C_∞ topology. Therefore $K(D_n \circ D_\infty^{-1})$ restricted to $D_\infty(V)$ converges to one. Compactness then implies that $K(f^n)$ converges to one where $f^n = f_n \circ f_\infty^{-1}$.

Let M^n be an exhaustion of M_∞ by compact sets with $\omega_\infty \in M^n$ for all n . The biLipshitz constants of f^n restricted to M^n may not limit to one. We need to relabel f^n and M^n for this to happen. Using the result for compact manifolds, we can define an strictly increasing function, $L : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, such that f^n restricted to M^m is a $(1+1/m)$ -biLipshitz map if $n > L(m)$. Now let $g^n = f^{n+L(1)}$ and let $X^n = M^m$ where $L(m) < n + L(1) \leq L(m+1)$. Then X^n exhaust M_∞ , while the g^n restricted to X^n have biLipshitz constant approaching one. 4.3

5 Torus ends

The main goal of this paper is to understand the space of hyperbolic structures on an open, tame 3-manifold M . To make this space tractable we need to restrict the possible geometries of the ends. In the next two sections we describe the allowable geometries of the ends.

Recall that an end, E , is a 3-manifold with boundary homeomorphic to $S \times [0, \infty)$ where S is a closed surface. As we will only be interested in the geometry of an end “near infinity” we make the following definition: Hyperbolic structures, E_1 and E_2 , on E are *isometric ends* if there exists a homeomorphism, $\phi : E_1 \rightarrow E_2$, that is an isometry outside of a compact subset of E_1 . By Theorem 4.2 if E'_1 is a small deformation of E_1 then there exists a small deformation E'_2 of E_2 such that E'_1 and E'_2 are isometric ends.

In this section we restrict to the case where $E = T \times [0, \infty)$ where T is a torus. Let $U = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ be the upper half plane of \mathbb{C} . For each $(a, b) \in \mathbb{C} \times U$ we define a $(PSL_2\mathbb{C}, \mathbb{H}^3)$ -developing map, $D_{(a,b)}$, in $\mathcal{D}(E)$. The hyperbolic structures defined by these developing maps are *cone structures* on E .

We parameterize the universal cover of E as $\mathbb{R}^2 \times [1, \infty)$. Let γ_1 and γ_2 be generators of $\pi_1(E) = \mathbb{Z}^2$. Then the action of $\pi_1(E)$ on the universal cover is defined by $\gamma_1(x, y, t) = (x + 1, y, t)$ and $\gamma_2(x, y, t) = (x, y + 1, t)$ for all $(x, y, t) \in \mathbb{R}^2 \times [1, \infty)$.

Let $s = (a, b)$ with $a \neq 0$. Before we describe D_s we define maps $\phi_s : \mathbb{R}^2 \rightarrow \mathbb{C}$ by

$$\phi_s(x, y) = -z_0 e^{xa+yab}$$

where $z_0 = \frac{1}{1-e^a}$. Then

$$D_s(x, y, t) = \frac{|\phi_s(x, y)|}{\sqrt{t^2 + |\phi_s(x, y)|^2}} (\phi_s(x, y), t) + (z_0, 0).$$

The corresponding holonomy representation, ρ_s , is defined on the generators, γ_1 and γ_2 , by $\rho_s(\gamma_1) = e^a z + 1$ and $\rho_s(\gamma_2) = e^{ab} z + \frac{e^{ab}-1}{e^a-1}$.

Now let $a = 0$ and define

$$D_s(x, y, t) = (x + by, t)$$

with holonomy representation $\rho_s(\gamma_1) = z + 1$ and $\rho_s(\gamma_2) = z + b$.

If $s_n = (a_n, b_n)$ with $a_n \rightarrow 0$ and $b_n \rightarrow b$ as $n \rightarrow \infty$ then the maps D_{s_n} converge to D_s in the C^∞ topology and the holonomy representations ρ_{s_n} converge to ρ_s .

We define E_s to be the corresponding hyperbolic cone-manifold and let $CT(E)$ be the space of ends, E_s . The topology of $\mathcal{D}(E)$ induces a topology on $CT(E)$.

Proposition 5.1 *The space $CT(E)$ is parameterized by $\mathbb{C}/\{\pm 1\} \times U$.*

Proof. If two hyperbolic structures, E_s and $E_{s'}$, are isometric then they must have conjugate holonomy representations. We separate these holonomy representations into two classes, parabolic representations and hyperbolic representations. The parabolic case is the easiest for ρ_s will be parabolic if and only if $s = (0, b)$ and this holonomy will be conjugate to $\rho_{s'}$ if and only if $s' = (0, b)$.

If ρ_s and $\rho_{s'}$ are hyperbolic then it is possible for ρ_s and $\rho_{s'}$ to be conjugate and E_s and $E_{s'}$ to not be isometric ends. To distinguish between E_s and $E_{s'}$ we examine the intrinsic geometry of the ends. A hyperbolic holonomy representation will fix an axis in \mathbb{H}^3 . Let T_ϵ be the torus in E_s whose image under the developing map is the locus of points distance ϵ from the axis of ρ_s . Similarly define T'_ϵ in $E_{s'}$. The ends E_s and $E_{s'}$ will be isometric if and only if there is an isometry between T_ϵ and T'_ϵ in the correct homotopy class. In particular, the length of γ_1 and γ_2 should be the same on both T_ϵ and T'_ϵ . These lengths can be easily calculated using hyperbolic trigonometric functions and it is seen that this will be the case if and only if $s = (a, b)$ and $s' = (\pm a, b)$. 5.1

We can now define a complex length functions,

$$\mathcal{L}_{x,y} : CT(E) \longrightarrow \mathbb{C}/\{\pm 1\},$$

for every $x, y \in \mathbb{R}$ by $\mathcal{L}_{x,y}(E_s) = \pm a(x + by)$. By Proposition 5.1, $\mathcal{L}_{x,y}$, is well defined. If $(x, y) = (p, q)$, for relatively prime p and q , then $\mathcal{L}_{p,q}(E_s)$ will be the length and rotation of the holonomy representation of the (p, q) curve in E_s .

Let $\mathcal{R}_0(T) \subset \mathcal{R}(T)$ be the subvariety of $\mathcal{R}(T)$ consisting of representations with infinite image and $R_0(T)$ be the quotient, $\mathcal{R}_0(T)/PSL_2\mathbb{C}$.

Proposition 5.2 *The holonomy map, $h : CT(E) \longrightarrow R_0(T)$, is a complex analytic local homeomorphism.*

Proof. It is well known that $R_0(T)$ is a 2-dimensional complex manifold. By our explicit description of ρ_s we see that h is holomorphic and locally injective. Since $CT(E)$ and $R_0(T)$ have the same dimension h must be a local homeomorphism. 5.2

Proposition 5.2 allows to us define length functions, $\mathcal{L}'_{x,y}$, in neighborhood of a representation $\rho \in R(E)$ by choosing a lift of this neighborhood in $CT(E)$. However, in this case the $\mathcal{L}'_{x,y}$ will not be unique and instead depends on the choice of lift. In practice, the geometric structure defining ρ will determine what lift to take.

The complex length functions determine a map

$$CT(E) \longrightarrow \mathbb{R}^2/\{\pm 1\} \cup \infty.$$

If $s = (a, b)$ and $a \neq 0$ then there is a unique $\pm(x, y) \in \mathbb{R}^2/\{\pm 1\}$ such that $\mathcal{L}_{x,y}(E_s) = \pm 2\pi i$. Then $\pm(x, y)$ are the *hyperbolic Dehn filling coordinates* for E_s . If $a = 0$ the Dehn filling coordinate for E_s is ∞ .

Let \bar{E}_s be the metric completion of E_s . The Dehn filling coordinate determines the topology of \bar{E}_s . There are three cases.

Case 1. If the Dehn filling coordinate for E_s is ∞ , then E_s is already complete so $\bar{E}_s = E_s$. We refer to E_s as a *cusp*.

Case 2. If the Dehn filling coordinates are (x, y) and there exists a $\theta \in \mathbb{R}^+$ such that $\frac{2\pi}{\theta}(x, y) = (p, q)$ where p and q are relatively prime integers, then \bar{E}_s is an open solid torus. The metric along the core curve of the solid torus is singular and the cross section of this curve is a hyperbolic cone with *cone angle*, θ . Then the curve (p, q) is the *meridian* of E_s and is trivial in the solid torus. In this case, we call E_s a *rational cone-manifold*. If the cone angle of E_s is 2π then metric extends to a smooth metric on the solid torus \bar{E}_s .

Case 3. If x/y is irrational then E_s is completed by the addition of a single point. In this case \bar{E}_s is not a manifold for the boundary of an ϵ -neighborhood of the additional point is a torus. In this case E_s is an *irrational cone-manifold*.

Lemma 5.3 *Let Z be a 1-dimensional complex submanifold of $CT(E)$ with $E_s \in Z$. If E_s is a cone-manifold with Dehn filling coordinates (x_0, y_0) , then assume $\mathcal{L}_{x_0, y_0}|_Z$ is injective at E_s . If E_s is a cusp assume there is some (x_0, y_0) such that $\mathcal{L}_{x_0, y_0}|_Z$ is injective at M_s . In either case, Z is parameterized by Dehn filling coordinates at E_s .*

Proof. We first show that the map to Dehn filling coordinates is continuous. This is clearly true away from ∞ . Assume the sequence, $\{E_{s_i}\}$, converges to a cusp, E_s , with $s_i = (\pm a_i, b_i)$. Let (x_i, y_i) be the Dehn filling coordinates for E_{s_i} . By definition $a_i(x_i + b_i y_i) = \pm 2\pi i$. Since $a_i \rightarrow 0$, we must have $(x_i + b_i y_i) \rightarrow \infty$. Furthermore, the b_i are bounded so x_i or y_i must go to infinity, proving continuity.

We now finish the proof if E_s is a cusp. Let A and B be coordinate functions for \bar{Z} , the lift of Z to $\mathbb{C} \times U$ with $(A(0), B(0)) = s$. The complex length functions, $\mathcal{L}_{x,y}$, lift to functions, $\tilde{\mathcal{L}}_{x_0,y_0} : (\mathbb{C} \times U) \rightarrow \mathbb{C}$ with $\tilde{\mathcal{L}}_{x_0,y_0}|_{\bar{Z}}$ injective at s . The derivative of $\tilde{\mathcal{L}}_{x,y}(A(z), B(z))$ at 0 is $A'(0)(x+yB(0))$. By assumption, when $(x, y) = (x_0, y_0)$ this derivative is non-zero so $A'(0) \neq 0$. If $(x, y) \neq (0, 0)$ then $(x + yB(0)) \neq 0$ because $\text{Im } B(0) > 0$. Therefore $\tilde{\mathcal{L}}_{x,y}|_{\bar{Z}}$ is injective at $(A(0), B(0))$ for all $(x, y) \neq (0, 0)$.

We now find a neighborhood, \bar{V} , of $(A(0), B(0))$ in \bar{Z} such that $\tilde{\mathcal{L}}_{x,y}$ is injective on \bar{V} for all (x, y) . From the compactness of the unit circle we can first find such a \bar{V} for all (x, y) on the unit circle. We then note that if $\tilde{\mathcal{L}}_{x,y}$ is injective on \bar{V} then $\tilde{\mathcal{L}}_{kx,ky}$ is also injective on \bar{V} if $k \neq 0$ so the same neighborhood, \bar{V} , works for all (x, y) .

If necessary we shrink \bar{V} so that it is a branched double cover of a neighborhood V in Z . The injectivity of the $\tilde{\mathcal{L}}_{x,y}$ on \bar{V} imply that the $\mathcal{L}_{x,y}$ are injective on V . Therefore $\mathcal{L}_{x,y}(E_t) = \pm 2\pi i$ for at most one E_t in V and $\mathcal{L}_{x,y}$ is zero only at E_s . This implies that the map to Dehn filling coefficients is injective. Since it is also continuous the lemma is proved in the cusped case.

The proof is essentially the same if E_s is a cone-manifold except the injectivity of $\tilde{\mathcal{L}}_{x_0,y_0}$ only implies that length functions with coefficients near (x_0, y_0) are injective. However, this is enough to prove the lemma. See [HK], Theorem 4.8, for a slicker version of this proof. 5.3

6 Higher genus ends

6.1 Geometrically finite ends

In §2.9 we defined the notion of geometric finiteness for ends of complete hyperbolic manifolds. We now generalize this definition so that it will apply to ends of cone-manifolds. Let $E = S \times [0, 1)$. We give the union, $E \cup S$ the product topology, $S \times [0, 1]$. A developing map, D , for E is *geometrically finite without rank one cusps* if D extends to a local homeomorphism, $\bar{D} : (\tilde{E} \cup \tilde{S}) \rightarrow (\mathbb{H}^3 \cup \hat{\mathbb{C}})$. The restriction of \bar{D} to $S \times \{1\}$ will be a $(PSL_2\mathbb{C}, \hat{\mathbb{C}})$ -developing map for S . A $(PSL_2\mathbb{C}, \hat{\mathbb{C}})$ -structure on S is a *projective structure*. If D is globally injective, the holonomy of D will be discrete and faithful and this definition of geometric finiteness agrees with that given in §2. We also remark that a projective structure on S determines a conformal structure. The restriction of \bar{D} to S defines a *projective structure at infinity* for E and the underlying conformal structure is the conformal structure at infinity

defined in §2.9. The following proposition is proved in [Br].

Proposition 6.1 *To hyperbolic structures E_1 and E_2 on E are isometric ends if and only if they have the same projective structure at infinity.*

Before we continue our discussion of geometrically finite ends we need to review some of the basic facts about projective structures on S . We denote the Teichmüller space of projective structures, $P(S)$. As mentioned above, a projective structure determines a complex structure, so there is a projection, $P(S) \longrightarrow T(S)$, where $T(S)$ is the Teichmüller space of complex structures discussed in §2.11. The fibers of this projection are the holomorphic quadratic differentials on each complex structure and therefore $P(S)$ has complex dimension $6g - 6$. We also have the following fundamental theorem:

Theorem 6.2 (Hejhal) *The holonomy map $P(S) \longrightarrow R(S)$ is a complex, local homeomorphism.*

We can realize a conformal structure, μ , on S as the quotient, U/Γ , where Γ is a Fuchsian group. If we use U as our topological model for \hat{S} with Γ the group of deck transformations then any projective structure on S with conformal structure μ is determined by a conformal developing map $f : U \longrightarrow \hat{\mathbb{C}}$. In this way the pair (f, Γ) determines a projective structure on S . Note that a projective structure does not determine a unique pair, (f, Γ) .

Remark. If (f', Γ') is another projective structure we cannot compare f to f' because although they have the same domain, U , this domain is acted on by a different group of deck transformations for each map. On the other hand if (f, Γ) and (f', Γ') are near each other in $P(S)$ then there will be a diffeomorphism, $\phi : U \longrightarrow U$, such that $f' \circ \phi$ is a developing map with deck transformations Γ and with f and $f' \circ \phi$ close in the C^∞ topology.

We now construct a geometrically finite end, $E_{(f, \Gamma)}$, that has projective structure at infinity (f, Γ) . To do so we will use a general method for extending a conformal map, $f : U \longrightarrow \hat{\mathbb{C}}$, to a map $\Theta_f : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$. For each $z \in U$ we let M_z^f be the *osculating* Möbius transformation, i.e. the unique element of $PSL_2\mathbb{C}$ whose two-jet agrees with f at z . Let $P \subset \mathbb{H}^3$ be the hyperbolic plane bounded by $\mathbb{R} \cup \infty$. Orient P such that the normal projection onto U is orientation preserving and let P_t be the image of the time t normal flow of P . For every $x \in \mathbb{H}^3$ there is a unique geodesic, r , orthogonal to P through x . Let $r(x)$ be the limit of this geodesic in U and define $\Theta_f(x) = M_{r(x)}^f(x)$.

Unfortunately Θ_f will not be a local homeomorphism on all of \mathbb{H}^3 . To examine when Θ_f is a local homeomorphism we will need a result of Epstein and Anderson. To state it we need the notion of the *Schwarzian derivative* of a locally injective holomorphic function,

$$SCf(z) = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2,$$

and its hyperbolic norm in U ,

$$\|SCf(z)\| = z^2 |SCf(z)|.$$

If f is conformal developing map for S , then a straightforward calculation shows that $\|SCf(z)\| = \|SCf(\gamma(z))\|$ for all $\gamma \in \pi_1(S)$.

Theorem 6.3 *Let $p \in P_d$, and e_3 the oriented unit normal vector for P_d at p . Then there exists a orthonormal basis e_1 and e_2 for the tangent space of P_d at p and an orthonormal basis e'_1 , e'_2 and e'_3 for the tangent space of \mathbb{H}^3 at $\Theta_f(p)$ such that the derivative of Θ_f at p in terms of these two bases is*

$$\begin{pmatrix} 1 + \frac{\|SCf(r(p))\|}{\cosh d} & 0 & 0 \\ 0 & 1 - \frac{\|SCf(r(p))\|}{\cosh d} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 1. In [And] what is actually calculated is the derivative of the composition of the map from U to P_d and Θ_f . However, since the map from U to P_d just multiplies the metric by $\cosh d$ our result follows immediately from the work in [And].

Remark 2. The Schwarzian derivative is a holomorphic quadratic differential. A quadratic differential defines a pair of singular foliations, the horizontal and vertical trajectories. The r_*e_1 will lie in the direction of the horizontal trajectory while r_*e_2 will lie in the direction of the vertical trajectory. The foliations will be singular were the Schwarzian is zero. However, in this case the choice of e_1 and e_2 is arbitrary.

We have the following corollary of Epstein and Anderson's result:

Corollary 6.4 *If f is equivariant, as above there exists a d_0 such that if $d > d_0$ and $x \in P_d$ then Θ_f is a local homeomorphism at x .*

Proof. We first note that Θ_f is equivariant. This follows immediately from the relationship $M_{r(\gamma(x))}^f = \rho_s(\gamma) \circ M_{r(x)}^f \circ \gamma^{-1}$ for all $x \in \mathbb{H}^3$

and $\gamma \in \pi_1(S)$. Furthermore since S is compact, equivariance also implies that $\|SCf(z)\|$ is bounded on U . We then choose d_0 such that $\cosh d_0 = \sup_{z \in U} \|SCf(z)\|$ proving the corollary. 6.4

Now choose d_0 as in Corollary 6.4 and let H_{d_0} be the subset of hyperbolic space bounded by P_{d_0} and U . Then the universal cover, \tilde{E} can be identified with H_{d_0} and the deck transformations acting on H_{d_0} will be Γ . Then Θ_f restricted to H_{d_0} is a developing map for E that defines a geometrically finite structure $E_{(f, \Gamma)}$ on E with projective structure at infinity (f, Γ) .

The following proposition shows that two geometrically finite ends are close if there projective structures at infinity are close.

Proposition 6.5 *Let $E_{(f_i, \Gamma_i)}$ be a sequence of geometrically finite ends such that (f_i, Γ_i) converges to $(f_\infty, \Gamma_\infty)$ in $P(S)$. Then there exists K_i -biLipshitz homeomorphisms, $f_i : E_{(f_\infty, \Gamma_\infty)} \rightarrow E_{(f_i, \Gamma_i)}$, with $K_i \rightarrow 1$.*

Proof. If $(f_i, \Gamma_i) \rightarrow (f_\infty, \Gamma_\infty)$ in $P(S)$ then we can choose the representatives for (f_i, Γ_i) such that $f_i \rightarrow f_\infty$ in the C^∞ topology and such that $\Gamma_i \rightarrow \Gamma_\infty$ in $\mathcal{R}(S)$. For each f_i let d_i be the constant given Corollary 6.4. Since $f_i \rightarrow f_\infty$, $d_i \rightarrow d_\infty$. Choose d' such that $d' > d_i$ for large i . To prove the proposition we construct developing maps, $D_i : H_{d'} \rightarrow \mathbb{H}^3$, with the fixed group of deck transformations, Γ_∞ , and show that $(D_\infty)_* \circ (D_i^{-1})_*$ converges uniformly to the identity in the hyperbolic metric.

Since $\Gamma_i \rightarrow \Gamma_\infty$ there exist conjugating diffeomorphisms $\phi_i : U \rightarrow U$ such that $\Gamma_i = \phi_i^{-1} \circ \Gamma_\infty \circ \phi_i$ and $\phi \rightarrow \text{id}$. We can extend the ϕ_i to maps $\Phi_i : \mathbb{H}^3 \rightarrow \mathbb{H}^3$. The map Φ_i is determined uniquely by the two conditions that Φ_i takes P_d to itself and $\phi_i(r(x)) = r(\Phi_i(x))$. For the hyperbolic metric on U , ϕ , is K'_i -biLipshitz with $K'_i \rightarrow 0$. Since P_d is a scaled version of the hyperbolic plane, Φ_i , will also be K'_i -biLipshitz and the derivatives, $(\Phi_i)_*$ will converge uniformly to the identity.

For the sequence of conformal map, f_i , the derivative of Θ_{f_i} will converge uniformly along a ray perpendicular to P by Theorem 6.3. Choose a fundamental domain for the action of Γ_∞ on U and let F be the pre-image of this domain under r . Then F is a compact family of rays perpendicular to P so $(\Theta_{f_i})_*$ will converge uniformly to $(\Theta_{f_\infty})_*$ on F .

Now define $D_\infty = \Theta_{f_\infty}|_{H_{d'}}$ and $D_i = \Theta_{f_i} \circ \Phi_i|_{H_{d'}}$. Then $(D_\infty)_* \circ (D_i^{-1})_*$ will converge uniformly on F and therefore by equivariance on all of $H_{d'}$. 6.5

We let $GF(E)$ denote the space of structures $E_{(f,\Gamma)}$. We have shown that every projective structure is the projective structure of a geometrically finite end. This fact along with Propositions 6.1 and 6.5 and Theorem 6.2 implies the following proposition:

Proposition 6.6 *The holonomy map $GF(E) \longrightarrow R(S)$ is locally onto.*

We close this section by defining a map, $b : GF(E) \longrightarrow T(S)$ by $b(E_{(f,\Gamma)}) = \mu$ where μ is the conformal structure of U/Γ . If V is a neighborhood of the holonomy representation, ρ , of $E_{(f,\Gamma)}$ such that $h|_V$ is injective then Theorem 6.2 allows us to define a map, $b' : V \longrightarrow T(S)$ by composing b with the local inverse of h . Since ρ will not be the holonomy of a unique structure in $GF(E)$, the map b' will depend on both ρ and $E_{(f,\Gamma)}$.

6.2 Geometrically infinite ends

If the hyperbolic structure on E is not geometrically finite, then it is not known in general if E has local deformations. We will examine a special case. If the developing map, D , for E is injective then the image of the holonomy of E will be a Kleinian group Γ . The quotient, $\mathbb{H}^3/\Gamma = M$, will be a hyperbolic 3-manifold homeomorphic to $S \times \mathbb{R}$ and E will isometrically embed as an end of M . Following our previous notation, we let $QD(\Gamma; S)$ be the space of quasiconformal deformations of Γ that fix the end-invariant of E . Now for each Kleinian group $\Gamma' \in QD(\Gamma; S)$, the quotient $\mathbb{H}^3/\Gamma' = M'$ will again be a hyperbolic structure on $S \times \mathbb{R}$. However, it is not immediate that the ends of M' are near the ends of M when Γ' is near Γ . To show this we need to extend a quasiconformal map of $\widehat{\mathbb{C}}$ to a biLipshitz map of \mathbb{H}^3 . The following result can be found in [Mc].

Theorem 6.7 *Let $\phi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be a k -quasiconformal homeomorphism. Then ϕ extends continuously to a K -biLipshitz map, $\Phi : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$, where $K = k^{3/2}$. Furthermore, this extension is natural in the sense that if γ and γ' are elements of $PSL_2\mathbb{C}$ such that*

$$\phi \circ \gamma' = \gamma \circ \phi$$

then

$$\Phi \circ \gamma' = \gamma \circ \Phi.$$

Given $\Gamma' \in QD(\Gamma; S)$ there is a k -quasiconformal map, $\phi : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$, that conjugates Γ to Γ' . Let Φ be the extension of ϕ given by Theorem 6.7. If we identify the universal cover, \tilde{E} , of E with a subspace of \mathbb{H}^3 such

that Γ acts as deck transformations then $\Phi|_{\tilde{E}}$ is a developing map that defines an end E' near E . Let $QD(E)$ be the space of ends constructed through such developing maps with the developing map topology. Theorem 6.7 implies that these developing maps will vary continuously in the space of all developing maps so we have:

Proposition 6.8 *The holonomy map, $QD(E) \longrightarrow QD(\Gamma; S)$, is a homeomorphism.*

7 The Dehn filling theorems

Let M be the interior of a compact, hyperbolizable 3-manifold N . Assume ∂N has n components consisting of tori, T_1, \dots, T_k , and higher genus surfaces, S_{k+1}, \dots, S_n . Given a compact core M_0 of M we label the components of $M \setminus M_0$, E_1, \dots, E_n . If $i \leq k$, then E_i will be homeomorphic to $T_i \times [0, 1)$ and if $i > k$ then E_i will be homeomorphic to $S_i \times [0, 1)$ where S_i has genus > 1 .

A hyperbolic structure on M is a *cone-manifold* if for $i \leq k$ there exists an $E_s \in CT(E_i)$ such that E_i and E_s are end isometric. M is a *rational cone-manifold* if all of these structures are rational or cusps. M is *geometrically finite without rank one cusps* if for $i > k$ there exists $E_{(f, \Gamma)} \in GF(E_i)$ such that E_i and $E_{(f, \Gamma)}$ are end isometric.

Let $\mathcal{GF}(M)$ be the space of developing maps that induce geometrically finite cone structures and let $GF(M)$ be the associated Teichmüller space. The topology on $\mathcal{GF}(M)$ induces a topology on $GF(M)$. To see this topology more explicitly we define a sub-basis for it. Let $M' \in GF(M)$ be a geometrically finite cone structure and let M'_0 be a compact core for M' . Then $M'(\epsilon)$ is the set of structures in $GF(M)$ such that there exists an $(1 + \epsilon)$ -biLipshitz embedding of M'_0 in the correct homotopy class. The sets $M'(\epsilon)$ form a sub-basis of neighborhoods for M' .

Now assume M is a rational cone manifold with holonomy representation ρ . If E_i , with $i \leq k$, is not a cusp, let (p_i, q_i) be the meridian. Otherwise choose (p_i, q_i) to be any simple closed curve on T_i . We then have length functions, $\mathcal{L}_{p_i, q_i} : CT(E_i) \longrightarrow \mathbb{C}/\{\pm 1\}$, and $\mathcal{L}'_{p_i, q_i} : V_i \longrightarrow \mathbb{C}/\{\pm 1\}$, where V_i is a neighborhood of the restriction of ρ to E_i .

If $i > k$ we have maps $b_i : GF(E_i) \longrightarrow T(S_i)$ and $b'_i : V_i \longrightarrow T(S_{g_i})$ where V_i is a neighborhood of the restriction of ρ to E_i . We combine these maps in the obvious way to get two maps, $\Phi : GF(M) \longrightarrow \mathbb{C}^k \times T(S_{k+1}) \times \dots \times T(S_n)$, and $\Phi' : V \longrightarrow \mathbb{C}^k \times T(S_{g_{k+1}}) \times \dots \times T(S_{g_n})$ where V is a neighborhood of ρ . The main result of [Br] is:

Theorem 7.1 *If M is a geometrically finite rational cone-manifold with holonomy representation, ρ , and all cone angles $\leq 2\pi$, then Φ' is a local homeomorphism at ρ .*

We now prove:

Theorem 7.2 *If M is a geometrically finite rational cone-manifold with all cone angles $\leq 2\pi$, then Φ is a local homeomorphism at M . Therefore $GF(M)$ is locally parameterized by hyperbolic Dehn filling coordinates and the conformal structure at infinity.*

Proof. Let $h : GF(M) \longrightarrow R(M)$ be the holonomy map. By the definition of Φ' , $\Phi = \Phi' \circ h$. To complete the proof we need to show that h is a local homeomorphism.

Let $D \in GF(M)$ be a developing map for M , $\rho = h(M)$ the holonomy and M_0 a compact core. If $\rho' \in R(M)$ is near ρ then by Theorem 4.1 there exist $D' \in \mathcal{D}(M_0)$ such that the holonomy of D' is ρ' and D' is near $D|_{M_0}$. By Theorem 4.2 D' extends to a developing map in $\mathcal{GF}(M)$ since by Proposition 5.2 and Proposition 6.6 for all nearby representations there is always a cone on E_i if $i \leq k$ and a geometrically finite structure without rank one cusps if $i > k$. Therefore h is onto.

Now assume that M^1 and M^2 are near M and that $h(M^1) = h(M^2)$. We again apply Theorem 4.1 to find a compact core M_0^1 of M^1 that isometrically embeds in M^2 . Since M_0^1 will extend to a unique geometrically finite cone structure this implies that $M^1 = M^2$ and h is injective.

Therefore h is a local homeomorphism and combining this fact with Theorem 7.1 implies that Φ is a local homeomorphism.

Lemma 5.3 allows us to replace the complex lengths with Dehn filling coefficients in the parameterization. 7.2

We can now prove Thurston's Dehn filling theorem, generalized to geometrically finite manifolds without rank one cusps. Before we state it we recall the notion of topological Dehn filling.

Let M be a compact 3-manifold whose boundary contains a torus T . Choose a basis for $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ so that an element of $\pi_1(T)$ is determined by a pair of integers (p, q) . For each relatively prime pair, (p, q) , there is a manifold, $N(p, q)$ and an embedding, $f_{p,q} : M \longrightarrow M(p, q)$, such that $M(p, q) - f_{p,q}(M)$ is a solid torus with boundary $f_{p,q}(T)$ and the image of the (p, q) curve in $N(p, q)$ is trivial. $N(p, q)$ is the (p, q) -Dehn filling of M . Let γ denote the core curve of the solid torus. If $M(p, q)$ has a hyperbolic

structure on its interior, then $M(p, q)$ is a *hyperbolic Dehn filling* of M if $f_{p,q}$ can be chosen such that $M(p, q) - f_{p,q}(M)$ is the geodesic representative of γ . Note that a hyperbolic structure on $M(p, q)$ may not be a hyperbolic Dehn filling of M if the geodesic representative of γ is not isotopic to γ . Also note that the holonomy representation ρ of $M(p, q)$ also defines a non-faithful, holonomy representation, $\hat{\rho}$, for M via the map $f_{p,q}$. We further note that if M also has a hyperbolic structure the map $f_{p,q}$ allows us to compare the end-invariant of M to that of $M(p, q)$.

If M has k torus boundary components, we can Dehn fill each of them. Let relatively prime integers, (p_i, q_i) , be the Dehn filling coefficients for the i -th torus and let $(\mathbf{p}, \mathbf{q}) = (p_1, q_1; \dots; p_k, q_k)$. Then $M(\mathbf{p}, \mathbf{q})$ is the (\mathbf{p}, \mathbf{q}) -Dehn filling of M .

Define $|\mathbf{p}, \mathbf{q}| = |p_1| + |q_1| + \dots + |p_k| + |q_k|$.

Theorem 7.3 *Let M be a compact 3-manifold with k torus boundary components and assume M has a geometrically finite hyperbolic structure without rank one cusps with holonomy ρ . We then have the following:*

1. *Except for a finite number of pairs for each i , for each collection of relatively prime pairs (\mathbf{p}, \mathbf{q}) there exist a geometrically finite hyperbolic (\mathbf{p}, \mathbf{q}) -Dehn filling $M(\mathbf{p}, \mathbf{q})$ of M with $M(\mathbf{p}, \mathbf{q})$ having the end-invariant as M .*
2. *$\rho_{\mathbf{p}, \mathbf{q}} \rightarrow \rho$ as $|\mathbf{p}, \mathbf{q}| \rightarrow \infty$.*
3. *If X is a submanifold of M such that the complement of M is a neighborhood of the cusps then $f_{\mathbf{p}, \mathbf{q}}|_X$ is $K_{\mathbf{p}, \mathbf{q}}$ -biLipshitz with $K_{\mathbf{p}, \mathbf{q}} \rightarrow 1$ as $\|\mathbf{p}, \mathbf{q}\| \rightarrow \infty$.*

Proof. (1) and (2) By Theorem 7.2 $GF(M)$ is locally parameterized by the conformal structures at infinity and hyperbolic Dehn filling coordinates. If the Dehn filling coordinates for a structure $M' \in GF(M)$ are a set of relatively prime integers, (\mathbf{p}, \mathbf{q}) , then the metric completion of M' is a smooth hyperbolic structure on $M(\mathbf{p}, \mathbf{q})$. A neighborhood of $\infty \times \dots \times \infty$ in $(\mathbb{R}^2 \cup \infty)^k$ contains all (\mathbf{p}, \mathbf{q}) that exclude a finite number of pairs for each i .

(3) If X were compact this would simply state that the $M_{\mathbf{p}, \mathbf{q}}$ converge geometrically to M which is Proposition 4.3. We are allowed to include the geometrically finite ends in X because of Proposition 6.5. 7.3

We next prove the Dehn filling theorem for the class of geometrically infinite 3-manifolds discussed in §3. For a manifold, M , in this class, Theorem 3.2 describes a local parameterization for the variety $R(M; \partial_1 M)$. There is a natural space of hyperbolic structures, $GF(M; \partial_1 M)$, on M associated to the representations in $R(M; \partial_1 M)$. As before, we fix the class of structures the ends of M can have. The isometry type of the torus and geometrically finite ends is defined as above. Each geometrically infinite end will be end isometric to an end in $QD(E_i)$. We then have a map, $\Phi : GF(M; \partial_1 M) \longrightarrow \mathbb{C}^k \times T(\partial_1 M)$. Following the proofs of Theorems 7.2 and 7.3 along with Proposition 6.8 we have:

Theorem 7.4 *Φ is a local homeomorphism at M . Furthermore (1) - (3) of Theorem 7.3 hold.*

8 A density theorem

Let $AH(M) \subset R(M)$ be the space of discrete faithful representations of $\pi_1(M)$ and $MP(M) \subset AH(M)$ those representations whose quotient manifold is geometrically finite without rank one cusps (often referred to as *minimally parabolic*). Expanding on a more restrictive conjecture of Bers, Thurston conjectured that $\overline{MP(M)} = AH(M)$.

If M has incompressible boundary, the density conjecture is a consequence of the ending lamination conjecture. Thurston's relative compactness theorems imply that every possible end invariant is realized in $\overline{MP(M)}$. The ending lamination conjecture implies that this structure is unique and hence $\overline{MP(M)} = AH(M)$.

In fact the ending lamination conjecture implies that there is a hierarchy of quasiconformal deformation spaces. Let Γ and Γ_0 be isomorphic Kleinian groups with ending laminations λ and λ_0 , respectively. If λ_0 is a sublamination of λ then the ending lamination conjecture implies that $QD(\Gamma) \subset \overline{QD(\Gamma_0)}$. The particular case that we will be interested in is when $\lambda - \lambda_0$ is a collection of isolated simple closed curves in λ . If λ is entirely simple closed curves then Γ is geometrically finite and the fact that $QD(\Gamma_0) \subset \overline{QD(\Gamma)}$ is a result of Abikoff [Ab]. In the following theorem we show that the isolated rank one cusps can be resolved in geometrically infinite manifolds.

Theorem 8.1 *Let Γ be a Kleinian group such that the geometrically infinite ends of $M = \mathbb{H}^3/\Gamma$ are incompressible and don't have any accidental*

parabolics. Let λ_0 and λ_1 be the union of the ending laminations for geometrically infinite and finite ends, respectively. Then there exists a Kleinian group, Γ_0 , with ending lamination, λ_0 , such that $QD(\Gamma) \subset \overline{QD(\Gamma_0)}$.

Proof. Let N be a compact topological manifold whose interior is homeomorphic to M . View λ_0 as a collection of curves in ∂N , define \mathcal{C} to be a collection of simple closed curves in N that are isotopic to the ending lamination λ_1 and let $\hat{N} = M - \mathcal{C}$. Since the curves in \mathcal{C} can be isotoped into ∂N there is isotopy of N inside of itself that “pushes” the ∂N past N . In other words there is a map $f : N \rightarrow \hat{N} \subset N$ which is a homeomorphism onto its image.

Using the Kleinian-Maskit combination theorem we can find parabolic isometries, $\gamma_1, \dots, \gamma_n$, such that $\mathbb{H}^3/\hat{\Gamma}$ is homeomorphic to \hat{N} with $\hat{\Gamma} = \Gamma * \gamma_1 * \dots * \gamma_n$. Furthermore we can choose these parabolics such that the image of $f_*(\pi_1(N))$ in $\hat{\Gamma}$ is Γ . Since there is also an inclusion of \hat{N} in N we can compare the end-invariants of \hat{N} to N . The geometrically infinite ends of \hat{N} will be isometric to the corresponding infinite ends of N .

Let $\hat{\rho}$ be the representation of $\pi_1(\hat{N})$ with image $\hat{\Gamma}$. Choose a basis for the tori $\partial\hat{N}$ associated to \mathcal{C} such that the $(1,0)$ -curve is trivial in N . Apply Theorem 7.4 to obtain manifolds M_n that are the $(\mathbf{1}, \mathbf{n})$ -hyperbolic Dehn filling of \hat{M} along \mathcal{C} . Let $f_n : \hat{M} \rightarrow M_n$ be the embeddings obtained from the theorem and $\hat{\rho}_n$ the representations of $\pi_1(\hat{N})$. By this construction the M_n will be homeomorphic to M and the maps $f_n \circ f$ will be homotopy equivalences. Let ρ_n be the representation of $\pi_1(N)$ induced by $f_n \circ f$. Since $\hat{\rho}_n \rightarrow \hat{\rho}$, $\rho_n \rightarrow \rho$ since ρ_n is the restriction of $\hat{\rho}_n$ to the subgroup $\pi_1(N)$.

Let Γ_0 be the image of ρ_n for some n . To show that $\Gamma \in \overline{QD(\Gamma_0)}$ we need to know that all the M_n are quasi-isometric. Since M_n and $M_{n'}$ are homeomorphic, they are quasi-isometric if the ends of M_n are quasi-isometric to the corresponding ends of $M_{n'}$. This is another consequence of Theorem 7.4. Therefore $\Gamma_n \in QD(\Gamma_0)$ and $\Gamma \in \overline{QD(\Gamma_0)}$.

Let $\Gamma' \in QD(\Gamma)$ and repeat the procedure. The ends of $M' = \mathbb{H}^3/\Gamma'$ will be quasi-isometric to the ends of M . Therefore the ends of M'_n will be quasi-isometric to the ends of M_n so all the Γ'_n will lie in $QD(\Gamma_0)$ and therefore $QD(\Gamma) \subset \overline{QD(\Gamma_0)}$. 8.1

We close with a question. In Theorems 7.4 and 8.1 we are forced to restrict to manifolds with geometrically infinite ends that do not have accidental parabolics. This is because we know that these ends have quasi-conformal deformations (Proposition 3.1) giving the necessary semi-stability property. On the other hand, geometrically finite ends are stable even if they

do not have quasiconformal deformations. Furthermore at a geometrically finite representation there is a half dimensional subspace of $R(S)$ where the end can be deformed and the conformal structure at infinity is fixed. By analogy, the ending lamination is a “degenerate” conformal structure so we ask if there is a half dimensional subspace of $R(S)$ where a infinite end can be deformed preserving the quasi-isometry type.

References

- [Ab] W. Abikoff. On boundaries of Teichmüller spaces and on Kleinian groups: III. *Acta Math.* **134**(1975), 211–237.
- [Ah] L. Ahlfors. Finitely generated Kleinian groups. *Amer. J. of Math.* **86**(1964), 413–429.
- [And] C. G. Anderson. Projective structures on Riemann surfaces and developing maps to \mathbb{H}^3 and $\mathbb{C}P^n$. *Preprint* (1999).
- [Bon] F. Bonahon. Bouts des variétés hyperboliques de dimension 3. *Annals of Math.* **124**(1986), 71–158.
- [BO] F. Bonahon and J. P. Otal. Variétés hyperboliques à géodésiques arbitrairement courtes. *Bull. London Math. Soc.* **20**(1988), 255–261.
- [Br] K. Bromberg. Rigidity of geometrically finite hyperbolic cone-manifolds. *preprint* (2000).
- [Can] R. D. Canary. A covering theorem for hyperbolic 3-manifolds and its applications. *Topology* **35**(1996), 751–778.
- [CEG] R. D. Canary, D. B. A. Epstein, and P. Green. Notes on notes of Thurston. In *Analytical and Geometric Aspects of Hyperbolic Space*, pages 3–92. Cambridge University Press, 1987.
- [Com] T.D. Comar. Hyperbolic Dehn surgery and convergence of Kleinian groups. *PhD thesis, University of Michigan* (1996).
- [CS] M. Culler and P. Shalen. Varieties of group representations and splittings of 3-manifolds. *Annals of Math.* **117**(1983), 109–146.
- [Gun] R. Gunning. *Lectures on Vector Bundles over Riemann Surfaces*. Princeton University Press, 1967.

- [HK] C. Hodgson and S. Kerckhoff. Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery. *J. Diff. Geom.* **48**(1998), 1–59.
- [Kap1] M. Kapovich. On the dynamics of pseudo-Anosov homeomorphisms of representation varieties of surface groups. *Ann. Acad. Sci. Fen.* **23**(1998), 83–100.
- [Kap2] M. Kapovich. Hyperbolic manifolds and discrete groups: Lectures on Thurston’s hyperbolization. *preprint* (1999).
- [Mc] C. McMullen. *Renormalization and 3-Manifolds Which Fiber Over the Circle*. Annals of Math. Studies 142, Princeton University Press, 1996.
- [PP] C. Petronio and J. Porti. Negatively oriented ideal triangulations and a proof of Thurston’s hyperbolic Dehn filling theorem. *Preprint*. (1999).
- [Sul] D. Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. In *Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference*. Annals of Math. Studies 97, Princeton, 1981.
- [Th] W. P. Thurston. *Geometry and Topology of Three-Manifolds*. Princeton lecture notes, 1979.
- [We1] A. Weil. On discrete subgroups of Lie groups. *Annals of Math.* **72**(1960), 369–384.
- [We2] A. Weil. Remarks on the cohomology of groups. *Annals of Math.* **80**(1964), 149–157.

DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF MICHIGAN
 EAST HALL, 525 E UNIVERSITY AVE
 ANN ARBOR, MI 48109